

Existence theorem on spectral function for singular nonsymmetric first order differential operators

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Abstract In this paper we study spectral function for a nonsymmetric differential operator on the half line. Two cases of the coefficient matrix are considered, and for each case we prove by Marchenko's method that, to the boundary value problem, there corresponds a spectral function related to which a Marchenko-Parseval equality and an expansion formula are established. Our results extend the classical spectral theory for self-adjoint Sturm-Liouville operators and Dirac operators.

Keywords Nonsymmetric first order differential operator, Spectral function, Expansion theorem

MSC 34L10, 47E05

1 Introduction

As a very essential mathematical problem, Weyl-Stone eigenfunction expansion [29, 32] in which the key role is spectral function for singular self-adjoint second order linear differential operators has been studied deeply by many renowned mathematicians such as K. Kodaira [14], N. Levinson [17], B. M. Levitan [18], E. C. Titchmarsh [30] and K. Yosida [34] and so on. This theory for the singular operators may be derived as a limiting case of the classical Sturm-Liouville expansion theorem for the regular operators, where the Parseval equality for the regular operators plays a very important role for the proofs. Similar ideas can also be applied to singular self-adjoint first order systems, for example, the Dirac operators [16, 19]. For general theory of eigenfunction expansion for self-adjoint and regular non-self-adjoint operators in Hilbert space, we refer to [2, 15, 22]. For multidimensional cases, see, e.g., [11]. Moreover, a two-fold spectral expansion in terms of principal functions of a Schrödinger

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operator has been derived in [1]. Recently, K. Kirsten and P. Loya [13] have obtained some interesting results on the spectral zeta function for a Schrödinger operator on the half line.

However, to the author's knowledge, for singular nonsymmetric differential operators, there are few results on eigenfunction expansion. The limiting approach for self-adjoint case can not be applied even for very simple case of nonsymmetric differential operators, since in general the corresponding regular spectrum has irregular behavior on the complex plane. In order to extend expansion theory to general case, V. A. Marchenko [20, 21] established an excellent method in dealing with the singular Sturm-Liouville operator with complex-valued potential. In this paper, inspired by the idea of V. A. Marchenko, we are going to establish expansion theorem in two cases for a singular nonsymmetric differential operator, where the key is to prove the existence of the corresponding spectral function. Our results can be extended to $2n \times 2n$ systems, and for simplicity we here will only consider the case of $n = 1$. For the regular case of this nonsymmetric differential operator, recently we have obtained some results on inverse spectral problems with applications to inverse problems for one-dimensional hyperbolic systems, see [24]–[27]. It is well known that for many differential operators there are intrinsic relations between their spectral functions and the corresponding Weyl functions (often called m -functions), and for the recent interesting results on Weyl functions see, e.g., [3, 6, 7, 12, 28, 35, 36]. For the asymptotic behavior of spectral functions for elliptic operators we refer to [8, 10, 23].

In this paper we consider boundary value problems generated by a nonsymmetric differential operator on the half line $0 \leq x < \infty$:

$$(A_P \varphi)(x) := B \frac{d\varphi}{dx}(x) + P(x)\varphi(x) = \lambda \varphi(x)$$

where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, both matrix-valued function $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in (C^1[0, \infty))^4$ and parameter λ are complex-valued. It is directly checked that the adjoint operator of A_P in some suitable Hilbert space is $-B \frac{d}{dx} + \overline{P^T}(x)$ and consequently A_P is nonsymmetric. Here and henceforth, \bar{c} denotes the complex conjugate of c and \cdot^T denotes the transpose of a vector or matrix under consideration. Here we point out that the spectrum problem for A_P with compact matrix-valued function P has been studied in [31].

In order to describe our results properly, we first give some information on distributions and we refer to [21] for more details. Let $\mathbb{K}^2(0, \infty)$ denote the set of all square integrable functions in $(0, \infty)$ with compact support. For $\sigma > 0$, we set $\mathbb{K}_\sigma^2(0, \infty) = \{f \in \mathbb{K}^2(0, \infty) : f(x) = 0 \text{ for } x > \sigma\}$. The entire function $e(\rho)$ is called the *function of exponential type* if $|e(\rho)| \leq C \exp(\sigma |\operatorname{Im} \rho|)$ where the positive constants C and σ depend on $e(\rho)$. Moreover, the index

$$\sigma_e = \overline{\lim}_{r \rightarrow \infty} r^{-1} \ln \left(\max_{|\rho|=r} |e(\rho)| \right)$$

is called the type of entire function $e(\rho)$. Let linear topological space Z be the set of all entire exponential type functions integrable on the real line. The sequence e_n converges to e in Z if $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |e_n(\rho) - e(\rho)| d\rho = 0$ and the types σ_n of the functions $e_n(\rho)$ are bounded: $\sup \sigma_n < \infty$. The set of all linear continuous functionals defined on the test space Z will be denoted by Z' whose components are called *distributions* (generalized functions). The sequence D_n converges to D in Z' if $\lim_{n \rightarrow \infty} \langle D_n, e(\rho) \rangle = \langle D, e(\rho) \rangle$ for all test functions $e \in Z$.

In this paper we consider two cases of the coefficient matrix P . The first case is special and will be described as follows. Let P be a continuously differentiable matrix-valued function satisfying $BP = PB$ and μ be a complex constant. Here it is easy to see that P is of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Consider the following boundary value problem

$$\begin{cases} B \frac{d\varphi}{dx}(x) + P(x)\varphi(x) = \lambda\varphi(x), & 0 < x < \infty, \\ \varphi(0) = \begin{pmatrix} \cosh \mu & \sinh \mu \\ \sinh \mu & \cosh \mu \end{pmatrix}. \end{cases} \quad (1.1)$$

Let $\varphi = \varphi(x, \lambda)$ be the solution to (1.1) and

$$\varphi_{[1]} = \begin{pmatrix} \varphi_{[1]}^{(1)} \\ \varphi_{[1]}^{(2)} \end{pmatrix} \text{ and } \varphi_{[2]} = \begin{pmatrix} \varphi_{[2]}^{(1)} \\ \varphi_{[2]}^{(2)} \end{pmatrix}$$

be the first and the second column vector of the matrix φ , i.e., $\varphi = (\varphi_{[1]} \ \varphi_{[2]})$. Similarly we denote the matrix inverse of φ by $\psi = \varphi^{-1} = (\psi_{[1]} \ \psi_{[2]})$. Now for

$$f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \in (L^2(0, \infty))^2, \quad g = \begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix} \in (L^2(0, \infty))^2$$

where $(L^2(0, \infty))^2$ denotes the product space of $L^2(0, \infty)$, we set

$$\omega_f^k(\rho) = \int_0^\infty f^T(x) \psi_{[k]}(x, i\rho) dx, \quad \eta_g^k(\rho) = \int_0^\infty \varphi_{[k]}^T(x, i\rho) g(x) dx \quad (k = 1, 2),$$

where $i = \sqrt{-1}$, $\rho \in \mathbb{R}$. Then we have the first main result of this paper.

Theorem 1. *It holds for the boundary value problem (1.1) that*

$$\int_0^\infty f^T(x) \overline{g(x)} dx = \frac{1}{2\pi} \sum_{k=1}^2 \int_{-\infty}^\infty \omega_f^k(\rho) \eta_g^k(\rho) d\rho. \quad (1.2)$$

Moreover, for $f \in (\mathbb{K}^2(0, \infty))^2$ with $\omega_f^k(\rho), \eta_g^k(\rho) \in Z$ ($k = 1, 2$), the following

expansion formula holds:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \sum_{k=1}^2 \int_{-\infty}^{\infty} \omega_f^k(\rho) \varphi_{[k]}(x, i\rho) d\rho \\ &= \frac{1}{2\pi} \sum_{k=1}^2 \int_{-\infty}^{\infty} \eta_f^k(\rho) \psi_{[k]}(x, i\rho) d\rho. \end{aligned} \quad (1.3)$$

We often call (1.2) (or (1.7)) the *Marchenko-Parseval equality* which means that a *spectral function* exists in corresponding boundary value problem. Historically, the concept of spectral function came from the classical theory of Weyl. Theorem 1 implies that $\frac{1}{2\pi}E$ is a spectral function corresponding to problem (1.1) with P satisfying $B\bar{P} = PB$, which is the same as the case of $P = 0$. Here and henceforth E denotes the 2×2 unit matrix.

For general matrix function $P \in (C^1[0, \infty))^4$ without the constraint $BP = PB$, we also can show the existence of the corresponding spectral function. More precisely, let Q be a 2×2 matrix satisfying $QB + BQ = B$ and $Q^2 = Q$. It is seen by simple computation that there exists matrix Q satisfying the above conditions, and the simplest one is $Q = \text{diag}(1, 0)$. It follows easily from $\det B = 1$ that $\det Q = 0$. Consider the following boundary value problems

$$\begin{cases} B \frac{d\varphi}{dx}(x) + P(x)\varphi(x) = \lambda\varphi(x), & 0 < x < \infty, \\ \varphi(0) = Q, \end{cases} \quad (1.4)$$

and

$$\begin{cases} -\frac{d\tilde{\varphi}}{dx}(x)B + \tilde{\varphi}(x)P(x) = \lambda\tilde{\varphi}(x), & 0 < x < \infty, \\ \tilde{\varphi}(0) = Q. \end{cases} \quad (1.5)$$

Denote the solutions to problems (1.4) and (1.5) by $\varphi(x, \lambda)$ and $\tilde{\varphi}(x, \lambda)$, respectively. For all 2×2 matrices $f, g \in (L^2(0, \infty))^4$, we set

$$\Phi_f(\rho) = \int_0^\infty f(x)\varphi(x, i\rho)dx, \quad \tilde{\Phi}_g(\rho) = \int_0^\infty \tilde{\varphi}(x, i\rho)g(x)dx, \quad (1.6)$$

where $i = \sqrt{-1}$, $\rho \in \mathbb{R}$. Then we have another main result of this paper.

Theorem 2. *To the problems (1.4) and (1.5) there corresponds a distribution-valued spectral function $D = (D_{kl})_{1 \leq k, l \leq 2}$ such that $D = QDQ$, $D_{kl} \in Z'$ and*

$$\int_0^\infty f(x)g(x)dx = \int_{-\infty}^\infty \Phi_f(\rho)D(\rho)\tilde{\Phi}_g(\rho)d\rho. \quad (1.7)$$

Moreover, for $f \in (\mathbb{K}^2(0, \infty))^4$ with $\Phi_f(\rho), \tilde{\Phi}_f(\rho) \in Z^4$, the following expansion formula holds:

$$f(x) = \int_{-\infty}^{\infty} \Phi_f(\rho) D(\rho) \tilde{\varphi}(x, i\rho) d\rho = \int_{-\infty}^{\infty} \varphi(x, i\rho) D(\rho) \tilde{\Phi}_f(\rho) d\rho. \quad (1.8)$$

Although Theorem 1 and Theorem 2 have shown the existence of spectral function for the singular nonsymmetric differential operator in two cases, we here point out that the uniqueness of spectral function for the operator does not hold generally, which is the same as that for Sturm-Liouville operators (see, e.g., [19]). Moreover, since the spectral function is distribution-valued, it is not a measure in general, which is different from the case of self-adjoint Sturm-Liouville operators. Besides, given singular nonsymmetric differential operators with general P , it is still an open problem to prove the existence of spectral functions under general boundary conditions. On the other hand, it is interesting to investigate the corresponding inverse problems, namely, given spectral functions or Weyl functions, find the differential operators. See [5] for the classical inverse problem to determine the potential of the Sturm-Liouville operator from its spectral function and [4] for determination of singular differential pencils from the Weyl function. Theorem 1 has implied that the uniqueness does not hold generally for the inverse problems, and we need impose other assumptions for uniqueness. In a forthcoming paper we will study the inverse problems for the singular nonsymmetric differential operator.

The paper is composed of four sections. In Section 2 we establish transformation formulae for our boundary value problems. Sections 3 and 4 are devoted to prove Theorem 1 and Theorem 2 by transformation formulae, respectively.

2 Transformation formulae

Set

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < x\}. \quad (2.1)$$

For $P_j = (P_{j,kl})_{1 \leq k,l \leq 2} \in (C^1[0, \infty))^4$ ($j = 1, 2$), we define

$$\theta_1(x) = \frac{1}{2} \int_0^x (P_{2,12} + P_{2,21} - P_{1,12} - P_{1,21})(s) ds \quad (2.2)$$

and

$$\theta_2(x) = \frac{1}{2} \int_0^x (P_{2,11} + P_{2,22} - P_{1,11} - P_{1,22})(s) ds. \quad (2.3)$$

Moreover let us put

$$R(P_1, P_2)(x) = \exp(-\theta_1(x)) \begin{pmatrix} \cosh \theta_2(x) & -\sinh \theta_2(x) \\ -\sinh \theta_2(x) & \cosh \theta_2(x) \end{pmatrix}. \quad (2.4)$$

Here we remark that $R(P_1, P_2)(0) = E$, $R(P_1, P_2)(x) = R^{-1}(P_2, P_1)(x)$ and $R\left(-\overline{P_1^T}, -\overline{P_2^T}\right)(x) = \overline{R(P_2, P_1)(x)}$. Let $M_2(\mathbb{C})$ be the set of all 2×2 complex-valued matrices. We first prove the following lemma.

Lemma 2.1. *For any $\lambda \in \mathbb{C}$, $Q \in M_2(\mathbb{C})$ with $\det Q = 0$ and $P_j \in (C^1[0, \infty))^4$ ($j = 1, 2$), let $\varphi_j = \varphi_j(x, \lambda)$ satisfy*

$$\begin{cases} B \frac{d\varphi_j(x)}{dx} + P_j(x)\varphi_j(x) = \lambda\varphi_j(x), & 0 < x < \infty, \\ \varphi_j(0) = Q. \end{cases} \quad (2.5)$$

Then there exists a unique $K(P_1, P_2; Q) = (K_{kl}(P_1, P_2; Q))_{1 \leq k, l \leq 2} \in (C^1(\overline{\Omega}))^4$ independent of λ such that for $0 \leq x < \infty$ and all $\lambda \in \mathbb{C}$

$$\varphi_2(x, \lambda) = R(P_1, P_2)(x)\varphi_1(x, \lambda) + \int_0^x K(P_1, P_2; Q)(x, y)\varphi_1(y, \lambda)dy. \quad (2.6)$$

(transformation formula)

Here $R(P_1, P_2)(x)$ is defined by (2.4).

Moreover, the kernel $K(P_1, P_2; Q)$ is the unique solution to the following problem of first order system (2.7)~(2.9):

$$\begin{aligned} B \frac{\partial K(P_1, P_2; Q)}{\partial x}(x, y) + \frac{\partial K(P_1, P_2; Q)}{\partial y}(x, y)B \\ + P_2(x)K(P_1, P_2; Q)(x, y) - K(P_1, P_2; Q)(x, y)P_1(y) = 0, \quad (x, y) \in \Omega. \end{aligned} \quad (2.7)$$

$$K(P_1, P_2; Q)(x, 0)BQ = 0 \quad (0 \leq x < \infty). \quad (2.8)$$

$$\begin{aligned} K(P_1, P_2; Q)(x, x)B - BK(P_1, P_2; Q)(x, x) \\ = B \frac{dR(P_1, P_2)}{dx}(x) + P_2(x)R(P_1, P_2)(x) - R(P_1, P_2)(x)P_1(x) \end{aligned} \quad (2.9)$$

$(0 \leq x < \infty).$

Proof. We prove this lemma by the idea used in [33]. Since $P_j \in (C^1[0, \infty))^4$ ($j = 1, 2$), it can be verified directly that, if $K(P_1, P_2; Q) \in (C^1(\overline{\Omega}))^4$ is the unique solution to problem (2.7)~(2.9), then (2.6) holds. Therefore, it is sufficient to prove the existence and the uniqueness of the solution to problem (2.7)~(2.9) for each $P_1, P_2 \in (C^1[0, \infty))^4$.

For clarity, we reduce the proof to a special case. By the condition $\det Q = 0$, we may assume that a complex constant c exists such that $q_2 = cq_1$ where q_1, q_2 are the first column vector and the second one of Q , respectively. Then it is sufficient to prove the existence and the uniqueness of the solution to problem (2.7)~(2.9) in the case $\varphi_j(0, \lambda) = q_1$, since problem (2.5) is linear. Moreover, since a complex constant c^* exists such that $q_1 = c^* \begin{pmatrix} \cosh \mu \\ \sinh \mu \end{pmatrix}$ where $\mu \in \mathbb{C}$,

it can be reduced to the case $\varphi_j(0, \lambda) = \begin{pmatrix} \cosh \mu \\ \sinh \mu \end{pmatrix}$. In this case, we denote the the solution to problem (2.7)~(2.9) by $K(P_1, P_2, \mu)(x, y)$, and (2.8) has the following form:

$$\begin{cases} K_{12}(P_1, P_2, \mu)(x, 0) = -\tanh \mu K_{11}(P_1, P_2, \mu)(x, 0), \\ K_{22}(P_1, P_2, \mu)(x, 0) = -\tanh \mu K_{21}(P_1, P_2, \mu)(x, 0). \end{cases} \quad (2.10)$$

If we set

$$\begin{cases} L_1(x, y) = K_{12}(P_1, P_2, \mu)(x, y) - K_{21}(P_1, P_2, \mu)(x, y), \\ L_2(x, y) = K_{11}(P_1, P_2, \mu)(x, y) - K_{22}(P_1, P_2, \mu)(x, y), \\ L_3(x, y) = K_{11}(P_1, P_2, \mu)(x, y) + K_{22}(P_1, P_2, \mu)(x, y), \\ L_4(x, y) = K_{12}(P_1, P_2, \mu)(x, y) + K_{21}(P_1, P_2, \mu)(x, y) \end{cases} \quad (2.11)$$

and $L = L(x, y) = (L_1(x, y), L_2(x, y), L_3(x, y), L_4(x, y))$, then we can rewrite (2.7)~(2.9) as follows:

$$\frac{\partial L_k(x, y)}{\partial x} - \frac{\partial L_k(x, y)}{\partial y} = f_k(x, y, L) \quad ((x, y) \in \Omega, k = 1, 2), \quad (2.12)$$

$$\frac{\partial L_k(x, y)}{\partial x} + \frac{\partial L_k(x, y)}{\partial y} = f_k(x, y, L) \quad ((x, y) \in \Omega, k = 3, 4), \quad (2.13)$$

$$L_k(x, x) = r_k(x) \quad (0 \leq x < \infty, k = 1, 2), \quad (2.14)$$

$$\begin{cases} L_3(x, 0) = \sinh(2\mu)L_1(x, 0) + \cosh(2\mu)L_2(x, 0) \\ L_4(x, 0) = -\cosh(2\mu)L_1(x, 0) - \sinh(2\mu)L_2(x, 0) \end{cases} \quad (0 \leq x < \infty), \quad (2.15)$$

where $f_k(x, y, L) = \frac{1}{2} \sum_{m=1}^4 (a_{km}(y) + b_{km}(x)) L_m(x, y)$ ($1 \leq k \leq 4$), here $a_{km}(y)$, $b_{km}(x)$ ($1 \leq k, m \leq 4$) are linear combinations of two elements of the matrix functions $P_1(y)$ and $P_2(x)$ respectively, and $r_k \in C^1[0, \infty)$ ($k = 1, 2$) are dependent only on P_1 and P_2 .

Integrating (2.12), (2.13) with (2.14) and (2.15) along the characteristics $x + y = \text{const.}$ and $x - y = \text{const.}$ respectively, we obtain the following integral equations:

$$L_k(x, y) = \int_y^{\frac{x+y}{2}} f_k(-s + x + y, s, L) ds + r_k\left(\frac{x+y}{2}\right) \quad ((x, y) \in \bar{\Omega}, k = 1, 2), \quad (2.16)$$

and

$$\begin{aligned} L_k(x, y) &= \int_0^y f_k(s + x - y, s, L) ds \\ &+ \int_0^{\frac{x-y}{2}} \{ \alpha_k f_1(-s + x - y, s, L) + \beta_k f_2(-s + x - y, s, L) \} ds \\ &+ \alpha_k r_1\left(\frac{x-y}{2}\right) + \beta_k r_2\left(\frac{x-y}{2}\right) \\ &((x, y) \in \bar{\Omega}, k = 3, 4), \end{aligned} \quad (2.17)$$

where $\alpha_3 = \sinh(2\mu)$, $\beta_3 = \cosh(2\mu)$ and $\alpha_4 = -\cosh(2\mu)$, $\beta_4 = -\sinh(2\mu)$.

The unique solution $L \in (C^1(\overline{\Omega}))^4$ to (2.16) and (2.17) can be obtained by the iteration method. In fact, setting

$$L_k^{(0)}(x, y) = 0 \quad ((x, y) \in \overline{\Omega}, 1 \leq k \leq 4),$$

$$L_k^{(n)}(x, y) = \int_y^{\frac{x+y}{2}} f_k \left(-s + x + y, s, L^{(n-1)} \right) ds + r_k \left(\frac{x+y}{2} \right) \\ ((x, y) \in \overline{\Omega}, n \geq 1, k = 1, 2),$$

and

$$L_k^{(n)}(x, y) \\ = \int_0^y f_k \left(s + x - y, s, L^{(n-1)} \right) ds \\ + \int_0^{\frac{x-y}{2}} \left\{ \alpha_k f_1 \left(-s + x - y, s, L^{(n-1)} \right) + \beta_k f_2 \left(-s + x - y, s, L^{(n-1)} \right) \right\} ds \\ + \alpha_k r_1 \left(\frac{x-y}{2} \right) + \beta_k r_2 \left(\frac{x-y}{2} \right) \\ ((x, y) \in \overline{\Omega}, n \geq 1, k = 3, 4),$$

we can obtain by induction the estimates for each $n \geq 1$

$$\left| L_k^{(n)}(x, y) - L_k^{(n-1)}(x, y) \right| \leq \omega(x) \frac{\zeta^{n-1}(x)}{(n-1)!} \quad ((x, y) \in \overline{\Omega}, 1 \leq k \leq 4), \quad (2.18)$$

where

$$\omega(x) = (|\sinh(2\mu)| + |\cosh(2\mu)| + 1) \max_{0 \leq s \leq x} (|r_1(s)| + |r_2(s)|)$$

and

$$\zeta(x) = (|\sinh(2\mu)| + |\cosh(2\mu)| + 1) x \max_{0 \leq s \leq x} \frac{1}{2} \sum_{k,l=1}^2 (|P_{1,kl}(s)| + |P_{2,kl}(s)|).$$

Thus $L_k(x, y) = \lim_{n \rightarrow \infty} L_k^{(n)}(x, y)$ ($1 \leq k \leq 4$) exist uniformly for $(x, y) \in \overline{\Omega}$ and we see that $L_k(x, y)$ ($1 \leq k \leq 4$) satisfy (2.16) and (2.17) with the bound $|L_k(x, y)| \leq \omega(x) \exp(\sigma(x))$.

Moreover, differentiating (2.16) and (2.17) with respect to x and y , we can similarly obtain by induction the following estimates

$$\left| \frac{\partial L_k^{(n)}(x, y)}{\partial x} - \frac{\partial L_k^{(n-1)}(x, y)}{\partial x} \right| \leq \xi(x) \frac{\zeta^{n-1}(x)}{(n-1)!} \quad ((x, y) \in \overline{\Omega}, 1 \leq k \leq 4), \quad (2.19)$$

$$\left| \frac{\partial L_k^{(n)}(x, y)}{\partial y} - \frac{\partial L_k^{(n-1)}(x, y)}{\partial y} \right| \leq \xi(x) \frac{\zeta^{n-1}(x)}{(n-1)!} \quad ((x, y) \in \overline{\Omega}, 1 \leq k \leq 4), \quad (2.20)$$

where

$$\begin{aligned} & \xi(x) \\ &= \frac{1}{2} (|\sinh(2\mu)| + |\cosh(2\mu)| + 1) \\ & \times \left\{ \max_{0 \leq s \leq x} (|r'_1(s)| + |r'_2(s)|) + \frac{1}{2} \omega(x) \exp(\zeta(x)) \right. \\ & \quad \left. \times \max_{0 \leq s \leq x} \sum_{k,l=1}^2 (|P_{1,kl}(s)| + |P_{2,kl}(s)| + (|P'_{1,kl}(s)| + |P'_{2,kl}(s)|) x) \right\}. \end{aligned}$$

Therefore, it follows from (2.19) and (2.20) that $L \in (C^1(\overline{\Omega}))^4$. The uniqueness of the solution to (2.7)~(2.9) is shown by (2.18). \square

Corollary 2.2. *For $j = 1, 2$, let φ_j be the solution to problem (1.1) with $P = P_j \in (C^1[0, \infty))^4$ satisfying $P_j B = B P_j$. Then the following transformation formula holds:*

$$\varphi_2(x, \lambda) = R(P_1, P_2)(x) \varphi_1(x, \lambda) \quad (2.21)$$

where $R(P_1, P_2)(x)$ is defined by (2.4).

Corollary 2.2 follows from the fact that $K(P_1, P_2, \mu) \equiv 0$, which can be derived easily by observing that the right hand side of (2.9) is 0 (in this case the condition $\det Q = 0$ is not necessary). Or one may directly verify (2.21). Here we omit the details.

Corollary 2.3.

Let S and \tilde{S} be the solutions corresponding to $P = 0$ in (1.4) and (1.5), respectively. Then the following transformation formulae hold.

(1) For problem (1.4) we have

$$S(x, i\rho) = R(P, 0)(x) \varphi(x, i\rho) + \int_0^x K(P, 0; Q)(x, y) \varphi(y, i\rho) dy \quad (2.22)$$

where the kernel $K(P, 0; Q) \in (C^1(\overline{\Omega}))^4$ satisfies the equation

$$BK_x(P, 0; Q)(x, y) + K_y(P, 0; Q)(x, y)B - K(P, 0; Q)(x, y)P(y) = 0, (x, y) \in \Omega \quad (2.23)$$

as well as the conditions for $0 \leq x < \infty$

$$K(P, 0; Q)(x, 0)Q = K(P, 0; Q)(x, 0), \quad (2.24)$$

and

$$K(P, 0; Q)(x, x)B - BK(P, 0; Q)(x, x) = BR'(P, 0)(x) - R(P, 0)(x)P(x). \quad (2.25)$$

(2) For problem (1.5) we have

$$\tilde{S}(x, i\rho) = \tilde{\varphi}(x, i\rho)R(0, P)(x) + \int_0^x \tilde{\varphi}(y, i\rho) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})(x, y)} dy \quad (2.26)$$

where the kernel $\overline{K^T(-\overline{P^T}, 0; \overline{Q^T})(x, y)}$ satisfies

$$Q \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})(x, 0)} = \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})(x, 0)}. \quad (2.27)$$

Proof. (1) is obvious, since $\det Q = 0$ and then Lemma 2.1 can be applied. Here (2.24) follows from (2.8), $BQ = B - QB$ and $B^2 = E$. Now we prove (2). Note that by (1.5) the function $\tilde{\varphi}(x, i\rho)$ satisfies

$$\begin{cases} B \frac{d\overline{\tilde{\varphi}^T}}{dx}(x) - \overline{P^T(x)\tilde{\varphi}^T(x)} = i\rho \overline{\tilde{\varphi}^T(x)}, & 0 < x < \infty, \\ \overline{\tilde{\varphi}^T(0)} = \overline{Q^T}. \end{cases}$$

Then one will obtain (2.26) by (1) if he notices the following fact: $R(0, P)(x) = \overline{R(-\overline{P^T}, 0)(x)}$. \square

Since the solutions to the boundary value problems with $P = 0$ are entire in λ , it follows easily from the transformation formulae that

Corollary 2.4. *For each fixed x , all solutions to the boundary value problems under consideration are entire in λ .*

3 Proof of Theorem 1

We divide the proof of Theorem 1 into four steps as follows.

First step. We first construct a regular spectral function. Let S denote the solution of (1.1) corresponding to $P = 0$. Set

$$\rho = -i\lambda \text{ and } \nu = -i\mu.$$

It is easy to see that

$$\begin{aligned} S = S(x, \lambda) &= \begin{pmatrix} \cosh(\lambda x + \mu) & \sinh(\lambda x + \mu) \\ \sinh(\lambda x + \mu) & \cosh(\lambda x + \mu) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\rho x + \nu) & i \sin(\rho x + \nu) \\ i \sin(\rho x + \nu) & \cos(\rho x + \nu) \end{pmatrix} \end{aligned} \quad (3.1)$$

and

$$S^{-1} = S^{-1}(x, \lambda) = \begin{pmatrix} \cos(\rho x + \nu) & -i \sin(\rho x + \nu) \\ -i \sin(\rho x + \nu) & \cos(\rho x + \nu) \end{pmatrix}. \quad (3.2)$$

We choose two sufficiently smooth real-valued functions $\delta_n(x)$ and $\gamma_\sigma(x)$ subject to the following conditions:

$$\begin{aligned} \int_0^\infty \delta_n(x) dx &= 1, \\ \delta_n(x) &= 0 \text{ for } x = 0 \text{ and } x \geq \frac{1}{n}, \quad \delta_n(x) > 0 \text{ for } 0 < x < \frac{1}{n}, \\ \gamma_\sigma(x) &= 1 \text{ for } 0 \leq x \leq \sigma, \quad \gamma_\sigma(x) = 0 \text{ for } x > \sigma + 1, \end{aligned} \quad (3.3)$$

and it is obvious that $\delta_n(x)$ tends to the Dirac delta function $\delta(x)$ as $n \rightarrow \infty$. We set

$$\begin{aligned} D_n^\sigma(\rho) &= (D_{n,jm}^\sigma(\rho))_{1 \leq j,m \leq 2} \\ &= \frac{1}{2\pi} \int_0^\infty \begin{pmatrix} \cos(\rho x + \nu) & -i \sin(\rho x + \nu) \\ -i \sin(\rho x + \nu) & \cos(\rho x + \nu) \end{pmatrix} \\ &\quad \times R(P, 0)(x) \delta_n(x) E \gamma_\sigma(x) \begin{pmatrix} \cos \nu & i \sin \nu \\ i \sin \nu & \cos \nu \end{pmatrix} dx. \end{aligned} \quad (3.4)$$

Since the Fourier transform is a one-to-one mapping on the space of bounded continuous Lebesgue-integrable functions and $R(P, 0)(x) \delta_n(x) E \gamma_\sigma(x)$ is a continuously differentiable matrix function with compact support, it is not hard to see that the matrix function $D_n^\sigma(\rho)$ is bounded and Lebesgue-integrable on the real line $-\infty < \rho < \infty$. Hence the integral

$$\int_{-\infty}^\infty S(x, i\rho) D_n^\sigma(\rho) \begin{pmatrix} \cos \nu & -i \sin \nu \\ -i \sin \nu & \cos \nu \end{pmatrix} d\rho$$

converges absolutely. By Corollary 2.2 we have $\varphi(x, i\rho) = R(0, P)(x) S(x, i\rho) = R^{-1}(P, 0)(x) S(x, i\rho)$, which implies by the Fourier inverse transform that

$$\int_{-\infty}^\infty \varphi(x, i\rho) D_n^\sigma(\rho) \begin{pmatrix} \cos \nu & -i \sin \nu \\ -i \sin \nu & \cos \nu \end{pmatrix} d\rho = \delta_n(x) E \quad (0 \leq x \leq \sigma). \quad (3.5)$$

Here and henceforth we repeatedly make use of the fact that two matrices P_1 and P_2 in the form of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are interchangeable: $P_1 P_2 = P_2 P_1$.

Second step. Next we will investigate the asymptotic behavior of the following matrix function as $n \rightarrow \infty$

$$\begin{aligned} U_n^\sigma(x, y) &= (U_{n,kl}^\sigma(x, y))_{1 \leq k,l \leq 2} \\ &:= \int_{-\infty}^\infty \varphi(x, i\rho) D_n^\sigma(\rho) \varphi^{-1}(y, i\rho) d\rho \quad (0 \leq x, y \leq \sigma). \end{aligned} \quad (3.6)$$

It is easy to find that

$$U_n^\sigma(x, 0) = \int_{-\infty}^{\infty} \varphi(x, i\rho) D_n^\sigma(\rho) \begin{pmatrix} \cos \nu & -i \sin \nu \\ -i \sin \nu & \cos \nu \end{pmatrix} d\rho = \delta_n(x) E \quad (0 \leq x \leq \sigma)$$

and

$$U_n^\sigma(0, y) = \int_{-\infty}^{\infty} \begin{pmatrix} \cos \nu & i \sin \nu \\ i \sin \nu & \cos \nu \end{pmatrix} D_n^\sigma(\rho) \varphi^{-1}(y, i\rho) d\rho \quad (0 \leq y \leq \sigma).$$

Now we show that $U_n^\sigma(0, y) = 0$ for all $y \geq 0$. Indeed, first one can see from (3.4) that

$$\begin{aligned} D_n^\sigma(\rho) &= \frac{1}{2\pi} \int_0^\infty \begin{pmatrix} \cos(\rho x) & -i \sin(\rho x) \\ -i \sin(\rho x) & \cos(\rho x) \end{pmatrix} \\ &\quad \times \{R_{11}(P, 0)(x)E + R_{12}(P, 0)(x)B\} \delta_n(x) \gamma_\sigma(x) dx. \end{aligned}$$

Moreover, for any continuous scalar function $u(x)$ with compact support and $u(0) = 0$, it follows easily from the theory of the Fourier cosine and sine transforms that

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_0^\infty \begin{pmatrix} \cos(\rho x) & -i \sin(\rho x) \\ -i \sin(\rho x) & \cos(\rho x) \end{pmatrix} u(x) \begin{pmatrix} \cos(\rho y) & -i \sin(\rho y) \\ -i \sin(\rho y) & \cos(\rho y) \end{pmatrix} dx d\rho \\ &= 0. \end{aligned} \tag{3.7}$$

Consequently, it follows from (3.7) and $\varphi^{-1}(\cdot, i\rho) = S^{-1}(\cdot, i\rho)R(P, 0)(\cdot)$ that $U_n^\sigma(0, y)R(0, P)(y) = 0$ and hence $U_n^\sigma(0, y) = 0$, since $R(0, P)(y)$ is invertible.

On the other hand, by (3.6) it is easy to see that, for fixed n and σ , $U_n^\sigma(\sigma, \cdot)$ is a bounded differentiable function on $[0, \sigma]$ and denoted by $\Xi_n(\cdot)$ for simplicity. Therefore, since by (1.1) we easily show that

$$\frac{d\varphi^{-1}(x)}{dx} B - \varphi^{-1}(x) P(x) = -i\rho \varphi^{-1}(x),$$

the above argument implies that the functions

$$U_{nN}^\sigma(x, y) := \int_{-N}^N \varphi(x, i\rho) D_n^\sigma(\rho) \varphi^{-1}(y, i\rho) d\rho \tag{3.8}$$

are continuously differentiable and satisfy the equation

$$B \frac{\partial U}{\partial x}(x, y) + \frac{\partial U}{\partial y}(x, y) B + P(x) U(x, y) - U(x, y) P(y) = 0 \text{ in } \Pi_\sigma \tag{3.9}$$

as well as the following conditions

$$U(x, 0) = \delta_{nN}(x) E \quad (0 \leq x \leq \sigma), \tag{3.10}$$

$$U(0, y) = \Gamma_{nN}(y), \quad U(\sigma, y) = \Xi_{nN}(y) \quad (0 \leq y \leq \sigma), \quad (3.11)$$

where $\Pi_\sigma = \{(x, y) \in \mathbb{R}^2 : 0 < x, y < \sigma\}$, the functions δ_{nN}, Γ_{nN} and Ξ_{nN} satisfy the compatibility conditions and $\lim_{N \rightarrow \infty} \delta_{nN}(x) = \delta_n(x)$, $\lim_{N \rightarrow \infty} \Gamma_{nN}(y) = 0$ and $\lim_{N \rightarrow \infty} \Xi_{nN}(y) = \Xi_n(y)$. We should note that problem (3.9), (3.10) and (3.11) can be rewritten as a symmetric hyperbolic system:

$$\begin{cases} \frac{\partial V}{\partial y}(x, y) + \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \frac{\partial V}{\partial x}(x, y) + C(x, y)V(x, y) = 0 \text{ in } \Pi_\sigma, \\ V(x, 0) = \delta_{nN}(x) \vec{H} \quad (0 \leq x \leq \sigma), \\ V(0, y) = \vec{\Gamma}_{nN}(y), \quad V(\sigma, y) = \vec{\Xi}_{nN}(y) \quad (0 \leq y \leq \sigma), \end{cases} \quad (3.12)$$

where

$$V(x, y) = \begin{pmatrix} U_{11}(x, y) \\ U_{12}(x, y) \\ U_{21}(x, y) \\ U_{22}(x, y) \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\vec{\Gamma}_{nN}(y) = \begin{pmatrix} \Gamma_{nN,11}(y) \\ \Gamma_{nN,12}(y) \\ \Gamma_{nN,21}(y) \\ \Gamma_{nN,22}(y) \end{pmatrix}, \quad \vec{\Xi}_{nN}(y) = \begin{pmatrix} \Xi_{nN,11}(y) \\ \Xi_{nN,12}(y) \\ \Xi_{nN,21}(y) \\ \Xi_{nN,22}(y) \end{pmatrix}$$

and $C(x, y)$ is the following 4×4 matrix-valued function

$$\begin{pmatrix} -P_{12}(y) & P_{11}(x) - P_{22}(y) & 0 & P_{12}(x) \\ P_{11}(x) - P_{11}(y) & -P_{21}(y) & P_{12}(x) & 0 \\ 0 & P_{21}(x) & -P_{12}(y) & P_{22}(x) - P_{22}(y) \\ P_{21}(x) & 0 & P_{22}(x) - P_{11}(y) & -P_{21}(y) \end{pmatrix}.$$

Since $BU_{nN}^\sigma(x, y) = U_{nN}^\sigma(x, y)B$, a direct calculation shows that the symmetric hyperbolic system (3.12) is actually equivalent to the following normal hyperbolic system

$$\begin{cases} \frac{\partial v}{\partial y}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial v}{\partial x}(x, y) + c(x, y)v(x, y) \text{ in } \Pi_\sigma, \\ v(x, 0) = \delta_{nN}(x) \vec{h} \quad (0 \leq x \leq \sigma), \\ v_2(0, y) = 2v_1(0, y) - \Gamma_{nN,11}(y) + 3\Gamma_{nN,12}(y), \\ v_2(\sigma, y) = 2v_1(\sigma, y) - \Xi_{nN,11}(y) + 3\Xi_{nN,12}(y) \quad (0 \leq y \leq \sigma), \end{cases} \quad (3.13)$$

where

$$v(x, y) = \begin{pmatrix} v_1(x, y) \\ v_2(x, y) \end{pmatrix} = \begin{pmatrix} U_{11}(x, y) - U_{12}(x, y) \\ U_{11}(x, y) + U_{12}(x, y) \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$c(x, y) = \begin{pmatrix} (P_{11} - P_{12})(x) + (P_{12} - P_{11})(y) & (P_{12} - P_{11})(x) + (P_{22} - P_{21})(y) \\ (P_{11} + P_{12})(y) - (P_{11} + P_{12})(x) & (P_{22} + P_{21})(y) - (P_{11} + P_{12})(x) \end{pmatrix}.$$

If we take the variable y as time, then it is not hard to verify that the classical Uniform Kreiss Condition holds, and hence from the well-known results of well-posedness on linear hyperbolic systems (cf. [9] and references therein) we see that (3.13) has a unique solution, that is, there exists a unique solution $U_{nN}^\sigma(x, y)$ to problem (3.9), (3.10) and (3.11) such that $U_{nN}^\sigma(x, y) \rightarrow U_n^\sigma(x, y)$ as $N \rightarrow \infty$.

On the other hand, if we set $W_{nN}^\sigma(x, y) = U_{nN}^\sigma(x, y) - \delta_{nN}(x - y)E$ for $0 \leq x, y \leq \sigma$ where $\delta_{nN}(x - y) = 0$ for $0 \leq x < y \leq \sigma$, then $W_{nN}^\sigma(x, y)$ satisfies the following equation

$$B \frac{\partial W}{\partial x}(x, y) + \frac{\partial W}{\partial y}(x, y)B + P(x)W(x, y) - W(x, y)P(y) \quad (3.14)$$

$$= \delta_{nN}(x - y)(P(y) - P(x))$$

and $W(x, 0) = 0$. It follows easily from the compatibility conditions that $W_{nN}^\sigma(0, y) \rightarrow 0$ ($N, n \rightarrow \infty$). Next we will show

$$W_{nN}^\sigma(\sigma, y) \rightarrow 0 \quad (N, n \rightarrow \infty). \quad (3.15)$$

In fact, it follows from (3.1), (3.2), (3.5) and the transformation formulae $\varphi(\cdot, i\rho) = R(0, P)(\cdot)S(\cdot, i\rho) = R^{-1}(P, 0)(\cdot)S(\cdot, i\rho)$, $\varphi^{-1}(\cdot, i\rho) = S^{-1}(\cdot, i\rho)R(P, 0)(\cdot)$ that

$$\begin{aligned} & \Xi_n(y) \\ &= R(0, P)(\sigma)R(P, 0)(y)R(P, 0)(\sigma - y) \\ & \quad \times \int_{-\infty}^{\infty} \varphi(\sigma - y, i\rho) D_n^\sigma(\rho) \begin{pmatrix} \cos \nu & -i \sin \nu \\ -i \sin \nu & \cos \nu \end{pmatrix} d\rho \\ &= R(0, P)(\sigma)R(P, 0)(y)R(P, 0)(\sigma - y)\delta_n(\sigma - y)E \quad (0 \leq x \leq \sigma). \end{aligned}$$

Thus, we have

$$\begin{aligned} & \Xi_n(y) - \delta_n(\sigma - y)E \\ &= \delta_n(\sigma - y)R(0, P)(\sigma)[R(P, 0)(y)R(P, 0)(\sigma - y) - R(P, 0)(\sigma)] \end{aligned} \quad (3.16)$$

whence (3.15) follows easily. Consequently, by the well-posedness of symmetric hyperbolic linear differential equations, we have

$$W_{nN}^\sigma(x, y) \rightarrow 0 \text{ as } N, n \rightarrow \infty$$

since $\delta_{nN}(x - y) \rightarrow \delta(x - y)$ as $N, n \rightarrow \infty$ and hence the right hand side of (3.14) tends to 0. Therefore, for $0 \leq x, y \leq \sigma$

$$U_n^\sigma(x, y) \rightarrow \delta(x - y)E \quad (n \rightarrow \infty). \quad (3.17)$$

Remark. There is another and simpler way to prove (3.17) in which it is not needed to consider (3.9). The key idea is based on considering (3.6), (3.7) and (3.16) with replacing σ by x . We leave the details to the reader.

Third step. We prove the Marchenko-Parseval equality (1.2). Assuming that $f, g \in (\mathbb{K}_\sigma^2(0, \infty))^2$ have compact support, we have by changing the order of integration that

$$\begin{aligned} & \int_0^\infty \int_0^\infty \sum_{k,l=1}^2 U_{n,kl}^\sigma(x, y) \overline{g^{(k)}(x)} f^{(l)}(y) dx dy \\ &= \int_0^\infty \int_0^\infty \sum_{k,l=1}^2 \left(\int_{-\infty}^\infty \sum_{j,m=1}^2 D_{n,jm}^\sigma(\rho) \varphi_{[j]}^{(k)}(x, i\rho) \psi_{[m]}^{(l)}(y, i\rho) d\rho \right) \\ & \quad \times \overline{g^{(k)}(x)} f^{(l)}(y) dx dy \\ &= \sum_{j,m=1}^2 \int_{-\infty}^\infty d\rho D_{n,jm}^\sigma(\rho) \\ & \quad \times \left(\int_0^\infty \int_0^\infty \sum_{k,l=1}^2 f^{(l)}(y) \psi_{[m]}^{(l)}(y, i\rho) \varphi_{[j]}^{(k)}(x, i\rho) \overline{g^{(k)}(x)} dx dy \right) \\ &= \sum_{j,m=1}^2 \int_{-\infty}^\infty D_{n,jm}^\sigma(\rho) \omega_f^m(\rho) \eta_g^j(\rho) d\rho. \end{aligned}$$

Therefore, in view of (3.17), we obtain by letting $n \rightarrow \infty$ that for any $f, g \in (\mathbb{K}_\sigma^2(0, \infty))^2$

$$\int_0^\infty f^T(x) \overline{g(x)} dx = \lim_{n \rightarrow \infty} \sum_{j,m=1}^2 \int_{-\infty}^\infty D_{n,jm}^\sigma(\rho) \omega_f^m(\rho) \eta_g^j(\rho) d\rho. \quad (3.18)$$

By the definition of $D_n^\sigma(\rho)$ (see (3.4)), we easily see that

$$\lim_{n \rightarrow \infty} D_n^\sigma(\rho) = \frac{1}{2\pi} R(P, 0)(0) = \frac{1}{2\pi} E. \quad (3.19)$$

On the other hand, since the Fourier transform is a continuous mapping of $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, it follows easily from Corollary 2.2 and the zero extensions of f and g on \mathbb{R} that both $\omega_f^m(\rho)$ and $\eta_g^j(\rho)$ belong to $L^2(\mathbb{R})$. Therefore, Combining (3.18) and (3.19) and letting $\sigma \rightarrow \infty$, we can assert (1.2) by the boundedness of $D_n^\sigma(\cdot)$, the dominated convergence theorem and the fact that $(\mathbb{K}^2(0, \infty))^2$ is dense in $(L^2(0, \infty))^2$.

Forth step. We prove the expansion (1.3). First we assume that $f \in (C_0[0, \infty))^2$, where $(C_0[0, \infty))^2$ denotes the product space of the set of all continuous functions with compact support. For any fixed real number $x \geq 0$ and $\delta > 0$, set

$$\varsigma(t) = \begin{cases} \frac{1}{\delta} & \text{for } t \in (x, x + \delta), \\ 0 & \text{for other case.} \end{cases} \quad (3.20)$$

In (1.2) first letting $g^{(1)}(t) = \varsigma(t)$, $g^{(2)}(t) = 0$ and then letting $g^{(1)}(t) = 0$, $g^{(2)}(t) = \varsigma(t)$, we have

$$\frac{1}{\delta} \int_x^{x+\delta} f(t) dt = \frac{1}{2\pi} \sum_{k=1}^2 \int_{-\infty}^{\infty} \omega_f^k(\rho) \frac{1}{\delta} \int_x^{x+\delta} \varphi_{[k]}(t, i\rho) dt d\rho.$$

Since

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_x^{x+\delta} f(t) dt = f(x)$$

and in Z

$$\lim_{\delta \rightarrow 0} \omega_f^k(\rho) \frac{1}{\delta} \int_x^{x+\delta} \varphi_{[k]}(t, i\rho) dt = \omega_f^k(\rho) \varphi_{[k]}(x, i\rho),$$

we prove the first part of (1.3) by the dominated convergence theorem if $f \in (C_0[0, \infty))^2$. For the case of $f \in (\mathbb{K}^2(0, \infty))^2$ we can approximate f by the functions in $(C_0[0, \infty))^2$. The second part of (1.3) can be proved similarly.

4 Proof of Theorem 2

First let us prove Theorem 2 for a special case. Recall that S and \tilde{S} are the solutions corresponding to $P = 0$ in (1.4) and (1.5), respectively.

Lemma 4.1. *For $f, g \in (L^2(0, \infty))^4$, it holds that*

$$\int_0^\infty f(x)g(x)dx = \frac{1}{\pi} \int_{-\infty}^\infty \Theta_f(\rho) \tilde{\Theta}_g(\rho) d\rho = \frac{1}{\pi} \int_{-\infty}^\infty \Theta_f(\rho) Q \tilde{\Theta}_g(\rho) d\rho$$

and for $x > 0$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^\infty \Theta_f(\rho) \tilde{S}(x, i\rho) d\rho = \frac{1}{\pi} \int_{-\infty}^\infty S(x, i\rho) \tilde{\Theta}_f(\rho) d\rho,$$

where $\Theta_f(\rho)$ and $\tilde{\Theta}_g(\rho)$ are defined by

$$\Theta_f(\rho) = \int_0^\infty f(x)S(x, i\rho)dx, \quad \tilde{\Theta}_g(\rho) = \int_0^\infty \tilde{S}(x, i\rho)g(x)dx. \quad (4.21)$$

Proof. Since it is easy to find that

$$S(x, i\rho) = Q \cosh(i\rho x) + BQ \sinh(i\rho x), \quad \tilde{S}(x, i\rho) = Q \cosh(i\rho x) - QB \sinh(i\rho x), \quad (4.22)$$

we have

$$\begin{aligned} \Theta_f(\rho) &= \int_0^\infty f(x)S(x, i\rho)dx = \frac{1}{2}\hat{f}(\rho)(Q - BQ) + \frac{1}{2}\hat{f}(-\rho)(Q + BQ), \\ \tilde{\Theta}_g(\rho) &= \int_0^\infty \tilde{S}(x, i\rho)g(x)dx = \frac{1}{2}(Q + QB)\hat{g}(\rho) + \frac{1}{2}(Q - QB)\hat{g}(-\rho), \end{aligned}$$

where $\hat{f}(\rho) = \int_0^\infty f(x)\exp(-i\rho x)dx$ denotes the Fourier transform of $f(x)$. Therefore, by the well-known Parseval equality

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(\rho)\hat{g}(-\rho)d\rho$$

and the identity for $u, v \in L^2(0, \infty)$

$$\int_{-\infty}^\infty \hat{u}(\rho)\hat{v}(\rho)d\rho = 0,$$

we easily obtain (note that $Q^2 = Q$)

$$\frac{1}{\pi} \int_{-\infty}^\infty \Theta_f(\rho)\tilde{\Theta}_g(\rho)d\rho = \frac{1}{\pi} \int_{-\infty}^\infty \Theta_f(\rho)Q\tilde{\Theta}_g(\rho)d\rho = \int_0^\infty f(x)g(x)dx.$$

On the other hand, since for all $u \in L^2(0, \infty)$ and $x > 0$ it holds that

$$\int_{-\infty}^\infty \hat{u}(\rho)\exp(-i\rho x)d\rho = \int_{-\infty}^\infty \hat{u}(-\rho)\exp(i\rho x)d\rho = 0,$$

we have

$$\begin{aligned} &\frac{1}{\pi} \int_{-\infty}^\infty \Theta_f(\rho)\tilde{S}(x, i\rho)d\rho \\ &= \frac{1}{2}f(x)(Q - BQ)(Q - QB) + \frac{1}{2}f(x)(Q + BQ)(Q + QB) = f(x). \end{aligned}$$

Similarly, we can show that $\frac{1}{\pi} \int_{-\infty}^\infty S(x, i\rho)\tilde{S}_f(\rho)d\rho = f(x)$. \square

If we put

$$f(x) = F(x)R(P, 0)(x) + \int_x^\infty F(t)K(P, 0; Q)(t, x)dt \quad (4.23)$$

and

$$g(x) = R(0, P)(x)G(x) + \int_x^\infty \overline{K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (t, x) G(t)} dt, \quad (4.24)$$

where F and G can be obtained by solving the above Volterra equations of the second kind, then it follows from changing the order of integration and the transformation formulae (2.22) and (2.26) that

$$\Phi_f(\rho) = \Theta_F(\rho), \quad \tilde{\Phi}_g(\rho) = \tilde{\Theta}_G(\rho). \quad (4.25)$$

Furthermore, we have

Lemma 4.2. *For $f, g \in (L^2(0, \infty))^4$, it holds that*

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F(x)G(x)dx + \int_0^\infty \int_0^\infty F(y)\mathfrak{F}(x, y)G(x)dxdy, \quad (4.26)$$

where $\mathfrak{F}(x, y)$ is defined as follows:

$$\mathfrak{F}(x, y) = \begin{cases} \overline{R(P, 0)(y)K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (x, y)} \\ \quad + \int_0^y K(P, 0; Q)(y, t) \overline{K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (x, t)} dt, & 0 \leq y \leq x, \\ \overline{K(P, 0; Q)(y, x)R(0, P)(x)} \\ \quad + \int_0^x K(P, 0; Q)(y, t) \overline{K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (x, t)} dt, & 0 \leq x \leq y. \end{cases} \quad (4.27)$$

Proof. On one hand, since $R(P, 0)(\cdot) = R^{-1}(0, P)(\cdot)$, we have by changing of the order of integration

$$\begin{aligned} & \int_0^\infty f(x)g(x)dx \\ &= \int_0^\infty \left\{ F(x)R(P, 0)(x) + \int_x^\infty F(t)K(P, 0; Q)(t, x)dt \right\} \\ & \quad \times \left\{ R(0, P)(x)G(x) + \int_x^\infty \overline{K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (t, x) G(t)} dt \right\} dx \\ &= \int_0^\infty F(x)G(x)dx + \int_0^\infty \int_x^\infty F(x)R(P, 0)(x) \overline{K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (t, x) G(t)} dt dx \\ & \quad + \int_0^\infty \int_0^t F(t)K(P, 0; Q)(t, x)R(0, P)(x)G(x)dxdt \\ & \quad + \int_0^\infty \int_x^\infty \int_x^\infty F(t)K(P, 0; Q)(t, x) \overline{K^T \left(-\overline{P^T}, 0; \overline{Q^T} \right) (s, x) G(s)} dt ds dx. \end{aligned}$$

On the other hand, by (4.27),

$$\begin{aligned}
& \int_0^\infty \int_0^\infty F(y) \mathfrak{F}(x, y) G(x) dx dy \\
&= \int_0^\infty \int_0^y F(y) \left\{ K(P, 0; Q)(y, x) R(0, P)(x) \right. \\
&\quad \left. + \int_0^x K(P, 0; Q)(y, t) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, t) dt \right\} G(x) dx dy \\
&\quad + \int_0^\infty \int_y^\infty F(y) \left\{ R(P, 0)(y) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, y) \right. \\
&\quad \left. + \int_0^y K(P, 0; Q)(y, t) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, t) dt \right\} G(x) dx dy.
\end{aligned}$$

Therefore, to prove (4.26), it is equivalent to show

$$\begin{aligned}
& \int_0^\infty \int_x^\infty \int_x^\infty F(t) K(P, 0; Q)(t, x) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(s, x) G(s) dt ds dx \\
&= \int_0^\infty \int_0^y \int_0^x F(y) K(P, 0; Q)(y, t) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, t) G(x) dt dx dy \\
&\quad + \int_0^\infty \int_y^\infty \int_0^y F(y) K(P, 0; Q)(y, t) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, t) G(x) dt dx dy,
\end{aligned}$$

which can be easily proved by changing of the order of integration. \square

Remark. It follows easily from (2.4) and Lemma 2.1 that $\mathfrak{F}(\cdot, \cdot) \in (C^1(\overline{\Omega}))^4$ and $\mathfrak{F}(\cdot, \cdot) \in (C^1(\mathbb{R}_+^2 \setminus \Omega))^4$.

Lemma 4.3. For $\mathfrak{F}(x, y)$ defined by (4.27), it holds that

$$\frac{\partial \mathfrak{F}}{\partial x}(x, y) B + B \frac{\partial \mathfrak{F}}{\partial y}(x, y) = 0 \tag{4.28}$$

and

$$\mathfrak{F}(x, 0) = \mathcal{J}(x), \quad \mathfrak{F}(0, y) = \mathcal{L}(y), \tag{4.29}$$

where

$$\mathcal{J}(x) = \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, 0), \quad \mathcal{L}(y) = K(P, 0; Q)(y, 0). \tag{4.30}$$

Moreover, the following relation holds:

$$\mathcal{J}(x) - B\mathcal{J}(x)B = \mathcal{L}(x) - B\mathcal{L}(x)B. \tag{4.31}$$

Proof. For $y \leq x$, in view of (2.7)~(2.9) in Lemma 2.1, we see by integration by parts that

$$\begin{aligned}
& \frac{\partial \mathfrak{F}}{\partial x}(x, y)B + B \frac{\partial \mathfrak{F}}{\partial y}(x, y) \\
&= \left\{ R(P, 0)(y) \overline{K_x^T(-\overline{P^T}, 0; \overline{Q^T})}(x, y) \right. \\
&\quad \left. + \int_0^y K(P, 0; Q)(y, s) \overline{K_x^T(-\overline{P^T}, 0; \overline{Q^T})}(x, s) ds \right\} B \\
&\quad + B \left\{ R'(P, 0)(y) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, y) \right. \\
&\quad \left. + K(P, 0; Q)(y, y) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, y) \right\} \\
&\quad + B \left\{ R(P, 0)(y) \overline{K_y^T(-\overline{P^T}, 0; \overline{Q^T})}(x, y) \right. \\
&\quad \left. + \int_0^y K_y(P, 0; Q)(y, s) \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, s) ds \right\} \\
&= \{ BR'(P, 0)(y) - R(P, 0)(y)P(y) + BK(P, 0; Q)(y, y) - K(P, 0; Q)(y, y)B \} \\
&\quad \times \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, y) + K(P, 0; Q)(y, 0) \overline{BK^T(-\overline{P^T}, 0; \overline{Q^T})}(x, 0) \\
&\quad + \int_0^y \{ BK_y(P, 0; Q)(y, s) + K_s(P, 0; Q)(y, s)B - K(P, 0; Q)(y, s)P(s) \} \\
&\quad \times \overline{K^T(-\overline{P^T}, 0; \overline{Q^T})}(x, s) ds \\
&= 0,
\end{aligned}$$

where we have made use of the relation: $B = QB + BQ$. For the case $x \leq y$, the proof of (4.28) is similar. On the other hand, (4.29) is obvious by (4.27).

Furthermore, it can be directly verified that the unique solution of problem (4.28) and (4.29) is

$$\mathfrak{F}(x, y) = \begin{cases} \frac{1}{2}\{\mathcal{J}(x+y) + \mathcal{J}(x-y)\} - \frac{1}{2}B\{\mathcal{J}(x+y) - \mathcal{J}(x-y)\}B, & y \leq x, \\ \frac{1}{2}\{\mathcal{L}(x+y) + \mathcal{L}(y-x)\} - \frac{1}{2}B\{\mathcal{L}(x+y) - \mathcal{L}(y-x)\}B, & x \leq y. \end{cases}$$

Consequently, (4.31) follows from the continuity of $\mathfrak{F}(x, y)$ at $x = y$. \square

Now we apply Lemma 4.3 to show

Lemma 4.4. *It holds that*

$$\Theta_{\mathcal{J}}(\rho)Q = \Theta_{\mathcal{J}}(\rho) = \tilde{\Theta}_{\mathcal{L}}(\rho) = Q\tilde{\Theta}_{\mathcal{L}}(\rho).$$

Proof. By (4.21), we have

$$\Theta_{\mathcal{J}}(\rho) = \int_0^\infty \mathcal{J}(x)S(x, i\rho)dx, \quad \tilde{\Theta}_{\mathcal{L}}(\rho) = \int_0^\infty \tilde{S}(x, i\rho)\mathcal{L}(x)dx,$$

where $S(x, i\rho)$ and $\tilde{S}(x, i\rho)$ are given by (4.22). Since $Q^2 = Q$, it is sufficient to prove that for all $x \geq 0$

$$\mathcal{J}(x)Q = Q\mathcal{L}(x), \quad \mathcal{J}(x)BQ = -QB\mathcal{L}(x).$$

First, multiplying right (4.31) by Q , we obtain by $QB + BQ = B$ and $\mathcal{L}(x)Q = \mathcal{L}(x)$ which follows from (2.24) and (4.30) that

$$\{\mathcal{J}(x) - B\mathcal{J}(x)B\}Q = \mathcal{L}(x)Q - B\mathcal{L}(x)(B - QB) = \mathcal{L}(x). \quad (4.32)$$

Second, since it follows from (2.27) and (4.30) that $Q\mathcal{J}(x) = \mathcal{J}(x)$, we have $QB\mathcal{J}(x) = (B - BQ)\mathcal{J}(x) = 0$. Consequently, it follows from (4.32) that

$$Q\mathcal{L}(x) = Q\{\mathcal{J}(x) - B\mathcal{J}(x)B\}Q = \mathcal{J}(x)Q - QB\mathcal{J}(x)BQ = \mathcal{J}(x)Q.$$

On the other hand, multiplying left (4.32) by B , we have by $B^2 = E$ that

$$B\mathcal{J}(x)Q - \mathcal{J}(x)BQ = B\mathcal{L}(x) = (QB + BQ)\mathcal{L}(x),$$

that is,

$$\mathcal{J}(x)BQ + QB\mathcal{L}(x) = B\{\mathcal{J}(x)Q - Q\mathcal{L}(x)\} = 0.$$

Thus the proof of Lemma 4.4 is completed. \square

Proof of Theorem 2. Let $\mathcal{L}_\sigma(x) = \gamma_\sigma(x)\mathcal{L}(x)$, $\mathcal{J}_\sigma(x) = \gamma_\sigma(x)\mathcal{J}(x)$, where the scalar function $\gamma_\sigma(x)$ is defined by (3.3). It is obvious that both \mathcal{L}_σ and \mathcal{J}_σ are continuously differentiable matrix-valued functions with compact support. Then it follows easily from Lemma 4.4 that $\Theta_{\mathcal{J}_\sigma}(\rho) = \tilde{\Theta}_{\mathcal{L}_\sigma}(\rho)$. Hence, combining (1.4), (1.5), (4.22), $Q\mathcal{J}(\cdot) = \mathcal{J}(\cdot)$, $\mathcal{L}(\cdot)Q = \mathcal{L}(\cdot)$ and Lemma 4.1, we conclude easily that the following matrix-valued function

$$\mathfrak{F}_\sigma(x, y) := \frac{1}{\pi} \int_{-\infty}^\infty S(y, i\rho)\Theta_{\mathcal{J}_\sigma}(\rho)\tilde{S}(x, i\rho)d\rho = \frac{1}{\pi} \int_{-\infty}^\infty S(y, i\rho)\tilde{\Theta}_{\mathcal{L}_\sigma}(\rho)\tilde{S}(x, i\rho)d\rho$$

satisfies the equation

$$U_x B + B U_y = 0,$$

and the conditions

$$U(x, 0) = \mathcal{J}_\sigma(x), \quad U(0, y) = \mathcal{L}_\sigma(y)$$

for all $x, y > 0$. Therefore, if we define $\mathfrak{F}_\sigma(0, 0) = \mathcal{L}(0) = \mathcal{J}(0)$, then $\mathfrak{F}_\sigma(x, y) = \mathfrak{F}(x, y)$ in the domain $0 \leq x, y \leq \sigma$, since $\gamma_\sigma(x) \equiv 1$ on $[0, \sigma]$ and the two matrix-valued functions satisfy the same boundary problem as that in Lemma 4.3. Moreover, if $f, g \in (\mathbb{K}_\sigma^2(0, \infty))^4$, then it follows from (4.23) and (4.24) that $F(x) = G(x) = 0$ for $x > \sigma$. Consequently, it follows from (4.21), (4.25), Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned}
& \int_0^\infty f(x)g(x)dx \\
&= \int_0^\infty F(x)G(x)dx + \int_0^\infty \int_0^\infty F(y)\mathfrak{F}(x, y)G(x)dx dy \\
&= \int_0^\infty F(x)G(x)dx + \int_0^\infty \int_0^\infty F(y)\mathfrak{F}_\sigma(x, y)G(x)dx dy \quad (4.33) \\
&= \frac{1}{\pi} \int_{-\infty}^\infty \Theta_F(\rho)\{Q + \Theta_{\mathcal{J}_\sigma}(\rho)\}\tilde{\Theta}_G(\rho)d\rho \\
&= \frac{1}{\pi} \int_{-\infty}^\infty \Phi_f(\rho)\{Q + \tilde{\Theta}_{\mathcal{L}_\sigma}(\rho)\}\tilde{\Phi}_g(\rho)d\rho.
\end{aligned}$$

Now define

$$D(\rho) = \lim_{\sigma \rightarrow \infty} \{Q + \Theta_{\mathcal{J}_\sigma}(\rho)\} = \lim_{\sigma \rightarrow \infty} \{Q + \tilde{\Theta}_{\mathcal{L}_\sigma}(\rho)\} \quad (4.34)$$

where the limits exist in the sense of convergence of distributions. Indeed, by (4.21) and (4.22) we see that both $\Theta_{\mathcal{J}_\sigma}(\rho)$ and $\tilde{\Theta}_{\mathcal{L}_\sigma}(\rho)$ are linear combination of the Fourier cosine and sine transform of some matrix-valued function with compact support. Then it follows from the property of the Fourier transform (see e.g. Page 105 in [21]) that $\Theta_{\mathcal{J}_\sigma}(\rho) \rightarrow \Theta_{\mathcal{J}}(\rho)$ and $\tilde{\Theta}_{\mathcal{L}_\sigma}(\rho) \rightarrow \tilde{\Theta}_{\mathcal{L}}(\rho)$ as $\sigma \rightarrow \infty$ in the sense of distributions, whence $D(\rho) \in (Z')^4$. Therefore, by the definition (4.21) we see

$$D(\rho) = \frac{1}{\pi} \{Q + \Theta_{\mathcal{J}}(\rho)\} = \frac{1}{\pi} \{Q + \tilde{\Theta}_{\mathcal{L}}(\rho)\}.$$

Thus we can prove the Marchenko-Parseval equality (1.7) similarly to (1.2). Moreover, if one lets $g(t) = \varsigma(t)E$ or $f(t) = \varsigma(t)E$ where $\varsigma(t)$ is defined by (3.20), then he can prove (1.8) similarly to (1.3). \square

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